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The derivation of low temperature expansions for the mixed spin Ising model

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Abstract. The derivation is discussed of low temperature (high field) expansions for the free energy per spin of the mixed spin Ising model. Both the direct method and the method of partial generating functions are considered. The lack of symmetry between the two sublattices considerably complicates the problem. Results are given, in the field grouping, for a number of two- and three-dimensional lattices.

1. Introduction

Mixed-spin Ising models (Schofield and Bowers 1980, 1981, Bowers and Schofield 1981, Yousif and Bowers 1983a, b, Bowers and Yousif 1983) are of interest for two main reasons. First, they have less translational symmetry than is usual—which is noteworthy in the light of the universality hypothesis. Second, they are well adapted for the investigation of a certain simple kind of ferrimagnetism (Néel 1948).

Before the present project began, series expansion studies of mixed spin Ising models were restricted to the high temperature regime (Schofield and Bowers 1981). Low temperature (high field) expansions for the standard 'single spin' models had, however, been available for some time. Much of the work was for spin-$\frac{1}{2}$ (Sykes *et al* 1965, 1973b) but this had also been extended to other spin values (Fox 1972, Sykes and Gaunt 1973).

This paper is concerned with the techniques which are necessary for the generation of low temperature (high field) expansions for mixed spin Ising models. Where possible the theory is given without restriction on the (two) spin magnitudes and in a style which generalises previous work—particularly that of Sykes and Gaunt (1973). Results are given for mixed spin-$\frac{1}{2}$/spin-1 Ising models on the honeycomb (HC), simple quadratic (SQ), simple cubic (SC) and body centred cubic (BCC) lattices. These have already found one published application—the study of the shape of the critical isotherm in mixed spin models (Yousif and Bowers 1983b).

Mixed spin Ising models have loose packed lattices of N sites. These lattices can be decomposed into two interpenetrating sublattices A and B in such a way that each site of A and q nearest neighbours which all belong to B (and *vice versa*). The sites of A and B are inhabited by spins of magnitude S_1 and S_2 respectively. The Hamiltonian

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takes the form

$$\mathscr{H} = -\frac{J}{S_1 S_2} \sum_{\langle ij \rangle} S_{1i}^z S_{2j}^z - \frac{m_A H_A}{S_1} \sum_i S_{1i}^z - \frac{m_B H_B}{S_2} \sum_j S_{2j}^z, \tag{1.1}$$

where each 'spin' S_{1i}^z takes the $2S_1+1$ values $-S_1, -S_1+1, \ldots, +S_1$ and each 'spin' S_{2j}^z takes the $2S_2+1$ values $-S_2, -S_2+1, \ldots, S_2$. In (1.1), the first summation is over all pairs of nearest neighbour sites in the lattice whilst the second and third summations are over all sites of the A and B sublattices respectively.

The quantities H_A, m_A and H_B, m_B are magnetic fields and magnetic moments per spin for the A and B sublattices, the notation being obvious. If the interaction constant J is positive, alignment of the spins is preferred and the situation is potentially ferromagnetic. If this constant is negative, the spins on one sublattice prefer to be antiparallel to the spins on the other and the situation is potentially ferrimagnetic (or perhaps, in the case $S_1 = S_2$, antiferromagnetic).

In the following sections of this paper, attention is restricted to the investigation of perturbation series which start from ground states in which all the spins on the lattice are parallel and 'fully up'. (Fully up spins have $S_{1i}^z = S_1$ or $S_{2j}^z = S_2$.) Each spin S_{1i}^z (S_{2j}^z) has accessible $2S_1$ ($2S_2$) excited states. A configuration of excited spins corresponds to a perturbed state of the lattice. The systematic consideration of such configurations—in the manner described below—yields perturbation series for thermodynamic properties.

The canonical distribution is employed and there are three external parameters—the temperature T and the two magnetic fields H_A and H_B. Perturbations are described using the variables

$$u = \exp[-\beta J/(S_1 S_2)], \qquad \mu = \exp(-\beta m_A H_A/S_1), \qquad \nu = \exp(-\beta m_B H_B/S_2), \tag{1.2}$$

where $\beta = 1/(kT)$ and k is Boltzmann's constant. The series are of the two standard types (Sykes *et al* 1973b). If one fixes the temperature—and therefore u—high field series in μ and ν can be obtained. This procedure can be followed irrespective of the sign of J. (If $J<0$, the ground state is still suitably ordered by sufficiently high unidirectional fields.) Series of the second type result when one fixes the fields H_A and H_B (at zero). Low temperature series in the variable u can then be obtained. These apply directly when $J>0$. A preferred direction ('up') is distinguished either using the fields H_A and H_B (which are later switched off) or, if that fails, arbitrarily. The ferromagnetic interactions then cause the spins to align in this direction in the ground state. Low temperature expansions can also be made relevant to the case $J<0$. This is achieved by transforming the ferrimagnet in a standard way (Bowers and Yousif 1983) to a ferromagnet with the same value of $|J|$ but with the field on one sublattice reversed. Low temperature expansions of this equivalent ferromagnet can then be employed.

2. Linkage rule

We start by developing a perturbation approach for the energy levels of (1.1).

The values accessible to spins S_{1i}^z and S_{2j}^z on the A and B sublattices are respectively $S_1 - x$ where $x = 0, 1, \ldots, 2S_1$ and $S_2 - y$ where $y = 0, 1, \ldots, 2S_2$. It is convenient to use the quantities x and y to label the states. Let N_x^A be the number of spins in the

xth state on the A sublattice and N_y^B be the number of spins in the yth state on the B sublattice. Let $N_{x,y}^{AB}$ be the number of (nearest neighbour) links of the lattice joining spins in the states x and y. The contribution to (1.1) from spin–spin interactions can be put in the form

$$E=-J\sum_{(x,y)} N_{x,y}^{AB}+\frac{J}{S_1S_2}\sum_{(x,y)}(S_1y+S_2x-xy)N_{x,y}^{AB}, \tag{2.1}$$

where the summations are over all ordered pairs (x, y).

The first summation in (2.1) is equal to the number of links on the lattice, which is $\frac{1}{2}qN$. Our intention is to rewrite the second summation so that all reference to the single spin ground states is removed. Each state x on A has q links attached to it. The total number of bonds attached to states x is therefore

$$qN_x^A=\sum_y N_{x,y}^{AB}. \tag{2.2}$$

Thus

$$\sum_x xqN_x^A=\sum_{(x,y)} xN_{x,y}^{AB} \tag{2.3}$$

and similarly

$$\sum_y yqN_y^B=\sum_{(x,y)} yN_{x,y}^{AB}. \tag{2.4}$$

Equations (2.3) and (2.4) allow (2.1) to be rewritten in the form

$$E=-\tfrac{1}{2}qJN+\frac{J}{S_1S_2}\left(qS_2\sum_{x>0} xN_x^A+qS_1\sum_{y>0} yN_y^B-\sum_{\substack{(x,y)\\ x,y\neq 0}} xyN_{x,y}^{AB}\right). \tag{2.5}$$

The contributions of the single spin ground states have been explicitly excluded in (2.5), simply because they vanish.

3. Series expansions

Each perturbed state has a corresponding Boltzmann factor

$$\exp\left[-\frac{\beta J}{S_1S_2}\left(S_2q\sum_{x>0} xN_x^A+S_1q\sum_{y>0} yN_y^B-\sum_{\substack{(x,y)\\ x,y\neq 0}} xyN_{x,y}^{AB}\right)\right.$$
$$\left.-\frac{\beta m_A H_A}{S_1}\sum_{x>0} xN_x^A-\frac{\beta m_B H_B}{S_2}\sum_{y>0} yN_y^B\right]. \tag{3.1}$$

This follows from (2.5) when the energy of interaction with the magnetic field (see (1.1)) is included.

An expansion for the (perturbative) partition function may be derived directly from (3.1). It takes the form

$$\Lambda_N(\mu,\nu,u)=\sum_{r,t,d}\Omega_N(r,t,d)u^{q(S_2r+S_1t)-d}\mu^r\nu^t, \tag{3.2}$$

where the variables are as at (1.2),

$$r = \sum_{x>0} xN_x^A, \qquad t = \sum_{y>0} yN_y^B, \qquad d = \sum_{\substack{(x,y)\\ x,y\neq 0}} xyN_{x,y}^{AB}, \tag{3.3}$$

and $\Omega_N(r, t, d)$ is the number of spin configurations with given r, t and d. It is useful to regard $\Omega_N(r, t, d)$ as a sum—over all values of N_x^A, N_y^B, $N_{x,y}^{AB}$ satisfying (3.3)—of the number of configurations with given N_x^A, N_y^B, $N_{x,y}^{AB}$. This yields a diagrammatic interpretation of the expansion and one sees that $\Omega_N(r, t, d)$ and (therefore) $\Lambda_N(r, t, d)$ are polynomials in N.

The expansion for the free energy per spin F may be obtained by generalising the method given by Domb (1960). One finds that

$$F = -\tfrac{1}{2}qJ - \tfrac{1}{2}(m_A H_A + m_B H_B) - kT \ln \Lambda(\mu, \nu, u), \tag{3.4}$$

where $\ln \Lambda(\mu, \nu, u)$ is the coefficient of N in (3.2). To deal successively with more and more excited spins, one writes

$$\ln \Lambda(\mu, \nu, u) = \sum_{r,t} g_{r,t}(u) \mu^r \nu^t. \tag{3.5}$$

The coefficients $g_{r,t}(u)$ are then polynomials since, from (3.2), one has

$$g_{r,t}(u) = \sum_d a[r, t, d] u^{q(S_2 r + S_1 t) - d}, \tag{3.6}$$

where $a[r, t, d]$ is the coefficient of N in the sum, over all possibilities consistent with (3.3), of the degeneracies of the states N_x^A, N_y^B, $N_{x,y}^{AB}$. These degeneracies are the numbers of embeddings of each given low temperature (high field) configuration which the lattice can sustain. One point is worth making. If q is odd and at least one of S_1 and S_2 is non-integral, then (3.5) is not a polynomial in u but in $z = u^{1/2}$.

The direct derivation of polynomials $g_{r,t}$ is, in principle, straightforward. By way of illustration the case $S_1 = \tfrac{1}{2}$, $S_2 = 1$ on the SQ lattice will be considered. For these spin values (on any lattice), there is only one perturbed state $x = 1$ possible on any site of A whilst there are two perturbed states $y = 1, 2$ possible on sites of B. For the present purposes it is convenient to fix attention on $g_{1,2}$. Here one has, from (3.3), the possibilities (i) $N_1^A = 1$, $N_1^B = 2$, $N_2^B = 0$ and (ii) $N_1^A = 1$, $N_1^B = 0$, $N_2^B = 1$. In case (i), there are (a) $\tfrac{1}{4}N(\tfrac{1}{2}N - 4)(\tfrac{1}{2}N - 5)$ configurations with $N_{1,1}^{AB} = 0$ (b) $2N(\tfrac{1}{2}N - 4)$ configurations with $N_{1,1}^{AB} = 1$, and (c) $3N$ configurations with $N_{1,1}^{AB} = 2$. (All these have $N_{1,2}^{AB} = 0$.) In case (ii), there are (a) $\tfrac{1}{2}N(\tfrac{1}{2}N - 4)$ configurations with $N_{1,2}^{AB} = 0$ and (b) $2N$ configurations with $N_{1,2}^{AB} = 1$. (All these have $N_{1,1}^{AB} = 0$.) If one takes the coefficient of N from each of the above and awards the appropriate power of u, one finds, from (3.4)–(3.6), that

$$g_{1,2}(u) = 5u^8 - 8u^7 + 3u^6 - 2u^8 + 2u^6 = 3u^8 - 8u^7 + 5u^6. \tag{3.7}$$

(The intermediate result is given to facilitate comparison with the preceding calculations.)

In the case where all the perturbed spins on A are in the same state x and all the perturbed spins on B are in the same state y, the combinatorial factors needed in calculations of the above type are the same as those for the spin-$\tfrac{1}{2}$ Ising antiferromagnet (Brooks and Domb 1951, Domb 1960). However, if x and/or y vary, the underlying configuration must be reweighted to allow for all possible decorations. This is already clear with a perturbation involving just two overturned spins on one sublattice. There are $\frac{1}{4}N(\frac{1}{2}N-1)$ configurations if the states are the same and $\frac{1}{2}N(\frac{1}{2}N-1)$ if they are different (the underlying configuration can then be decorated in two ways).

A systematic approach of the above type yields a useful number of $g_{r,t}$. Polynomials for smaller r and t were first calculated in this way. However, the final—more extensive—calculations were done using the procedure outlined in § 4 which exploits the sublattice division more effectively.

4. Methods of partial generating functions

For the present problem one can define partial generating functions which are equivalent to the perturbative free energy, when the number of excited spins on one sublattice—but not on the other—is held fixed. The method is a generalisation of that available for loose packed 'single spin' Ising models for which there is a well developed theory not only for spin-$\frac{1}{2}$ (Sykes *et al* 1965) but also for higher spin (Sykes and Gaunt 1973). It is convenient to describe the technique by continuing with the example of $S_1=\frac{1}{2}$, $S_2=1$, and the SQ lattice.

It helps to be concrete, so it will be supposed, first, that there are two overturned spins on the A (i.e. spin-$\frac{1}{2}$) sublattice. Now, quite generally, an excited spin casts a shadow—which in the present case is square—on its q nearest neighbours on the other sublattice. A configuration of excited spins on one sublattice will in this way give rise to a set of, possibly overlapping, shadows. To return to the example, here, the two shadows can overlap along an edge, at a vertex, or not at all depending on whether they are cast by spins which are *in A*, respectively, nearest neighbours, next nearest neighbours, or neither of these. The result is that six, seven or eight spins of the B sublattice are, respectively, in shadow.

Any site of the B sublattice which is perturbed yields one link for each shadow in which it lies. With this in mind one can construct the appropriate partial generating function (PGF). Consider first, with Sykes and Gaunt (1973), the case in which B is also a spin-$\frac{1}{2}$ sublattice. The total contribution to Λ_N is then

$$Ny^2(1+bx)^4(1+b^2x)^2(1+x)^{N/2-6}+Ny^2(1+bx)^6(1+b^2x)(1+x)^{N/2-7} + \tfrac{1}{4}N(\tfrac{1}{2}N-9)y^2(1+bx)^8(1+x)^{N/2-8}. \qquad (4.1)$$

Here, each term represents the total contribution to Λ_N from a given configuration of two overturned spins on A and all possible spin states on B (which is, temporarily, spin-$\frac{1}{2}$). The first factor in each term in (4.1) is the number of ways each two-spin pattern can be embedded in the lattice. The variables in (4.1) are obtained by comparison with (3.2). Thus

$$y=\mu u^{qS_2}, \qquad x=\nu u^{qS_1}, \qquad b=u^{-1}, \qquad (4.2)$$

and each overturned A spin yields y, each overturned B spin x, and each link or bond u^{-1}. (There should be no confusion between this and the previous use of x and y.)

To make (4.1) and (4.2) clear, it should suffice to explain the first term. This corresponds to the case in which the shadows overlap along an edge. There are N such configurations. The four spins belonging to only one shadow each contribute a factor $1+bx$: the 1 when unperturbed, the bx when overturned. The two spins on the common edge similarly yield $1+b^2x$ each. The $\frac{1}{2}N-6$ spins outside the shadow each yield $1+x$ (they can make no excited links).

The contribution to $\ln \Lambda$ corresponding to (4.1)—the PGF—is, as explained in § 3, the coefficient of N in the given expression. This is

$$y^2(1+bx)^4(1+b^2x)^2(1+x)^{-6}+y^2(1+bx)^6(1+b^2x)(1+x)^{-7}-\tfrac{9}{4}y^2(1+bx)^8(1+x)^{-8}. \tag{4.3}$$

The connection between 'partial partition functions' and partial generating functions is always of this simple form and quantities of the second type are usually written directly. The nth PGF is a sum, over all possible shadow patterns, of terms each of the form

$$y^nG\prod_m(1+b_x^m)^{\alpha_m}(1+x)^{-\Sigma_m\alpha_m}. \tag{4.4}$$

(We find it convenient to include terms such as y^n in our definitions.) In (4.4), G is the part linear in N in the number of ways of embedding the given shadow pattern and α_m is the number of sites in B which belong to exactly m (single A spin) shadows. (The quantity m is limited by the coordination number of the lattice.)

It is now easy to describe what happens when B reverts to spin 1. Then x (4.2) corresponds to a spin in the first excited state and one needs to add

$$X=v^2u^{2qS_1} \tag{4.5}$$

(see (3.2), (3.3)) to denote a spin in the second excited state. Furthermore, according to the above linkage rule bonds between y and X must carry a contribution $b^2=u^{-2}$. Since there are now three states on each B site, all this means that (4.3) becomes

$$y^2(1+bx+b^2X)^4(1+b^2x+b^4X)^2(1+x+X)^{-6}+y^2(1+bx+b^2X)^6(1+b^2x+b^4X)\times(1+x+X)^{-7}-\tfrac{9}{4}y^2(1+bx+b^2X)^8(1+x+X)^{-8}. \tag{4.6}$$

It is clear that, given any PGF for the spin-$\frac{1}{2}$ Ising model, a corresponding one for the present mixed spin model can be obtained by replacing each term of the form (4.4) by

$$y^nG\prod_m(1+b^mx+b^{2m}X)^{\alpha_m}(1+x+X)^{-\Sigma\alpha_m}. \tag{4.7}$$

In this way, using published results for the spin-$\frac{1}{2}$ Ising model (Sykes *et al* 1965, 1973a, c), what will be called type A partial generating functions for our mixed spin model have been constructed (Yousif 1983).

Thus far only the case in which the number of excited spins on the spin-$\frac{1}{2}$ (or A) sublattice is held fixed has been discussed. The other case can be investigated using the same variables and very similar techniques. Suppose two spin-1 objects on the B sublattice of a SQ lattice are excited. Then they give rise to the same shadow figures as previously but now the single spin square shadows must be decorated in all possible ways with x and X. This leads to each contribution at (4.3) splitting into three. Each term y^2 must be replaced successively by x^2, $2xX$ and X^2. Also each factor involving b and x in (4.3) must be replaced by new factors involving b and y. These vary even amongst each set of three 'similar' contributions since different patterns of xy and Xy bonds are found. The PGF corresponding to (4.1) can be calculated directly as described.

The result is

$$x^2(1+by)^4(1+b^2y)^2(1+y)^{-6}+2xX(1+by)^2(1+b^2y)^2(1+b^3y)^2(1+y)^{-6}$$
$$+X^2(1+b^2y)^4(1+b^4y)^2(1+y)^{-6}+x^2(1+by)^6(1+b^2y)(1+y)^{-7}$$
$$+2xX(1+by)^3(1+b^2y)^3(1+b^3y)(1+y)^{-7}$$
$$+X^2(1+b^2y)^6(1+b^4y)(1+y)^{-7}$$
$$-\tfrac{9}{4}x^2(1+by)^8(1+y)^{-8}-\tfrac{9}{2}xX(1+by)^4(1+b^2y)^4(1+y)^{-8}$$
$$-\tfrac{9}{4}X^2(1+b^2y)^8(1+y)^{-8}. \tag{4.8}$$

In this hierarchy (4.4) is replaced not by (4.7) but by

$$x^\beta X^\gamma G\tau \prod_{m,l}(1+b^{m+2l}y)^{\alpha_{m,l}}(1+y)^{-\Sigma_{m,l}\alpha_{m,l}}, \tag{4.9}$$

which is to be summed over all possible decorated shadow patterns. In (4.9), $\alpha_{m,l}$ is the number of sites in A which belong to the shadow of m x spins and l X spins and τ is the symmetry number of the decoration (1 for x^2 and X^2 and 2 for xX in the example). A few PGFs of this second kind (type B) have been calculated on the four lattices of interest (Yousif 1983). (It is already clear from (4.8) that the calculations soon become rather heavy.) The known spin-$\frac{1}{2}$ Ising results again provide a useful starting point.

5. High field polynomials

The high field polynomials $g_{r,t}$ can be extracted from the PGFs in various ways. The basic idea is simple. Given PGFs of one type (say A) to order n_A, exact information on *all* spin perturbations of up to n_A spins is available. This is because the information is exact to order n_A on one sublattice and to all orders on the other. Thus, by expanding PGF 0 to degree n_A in x and X, PGF 1 to degree $n_A-1, \ldots,$ PGF n_A to degree 0 and making the substitutions (4.2) and (4.5), one obtains more than enough information to determine all the coefficient polynomials $g_{r,t}$ with $r+t\leq n_A$. (There is more than enough information since each factor X becomes ν^2.)

If PGFs of both types are simultaneously available, further progress can be made. Suppose that one has type A PGFs to order n_A and type B to order n_B. Then one has exact information on *all* spin perturbations of up to n_A+n_B+1 spins. This can be extracted by the following procedure. For type A, expand PGF 0 to degree n_A+n_B+1 in x and X, PGF 1 to degree $n_A+n_B, \ldots,$ PGF n_A to degree n_B+1. For type B, expand PGF 0 to degree n_A+n_B+1 in y, PGF 1 to degree $n_A+n_B, \ldots,$ PGF n_B to degree n_A+1. Substitute for x, X, y and b from (4.2) and (4.5) and pick out the coefficient polynomials $g_{r,t}$ with $r+t\leq n_A+n_B+1$. (One will again have more than enough information since X is replaced by ν^2.) This procedure will cover spin perturbations of not more than n_A A spins and n_B B spins twice and hence provide a useful check. Spin perturbations outside this region—but involving not more than n_A+n_B+1 spins in total—will be covered once: by type A PGFs if n_B is exceeded on the B sublattice and *vice versa*.

Procedures of the above kind have been used for all the four lattices of interest here. In this way all high field polynomials $g_{r,t}$ with $r+t\leq 7$ have been found on the

SQ, SC and BCC lattices. On the HC lattice rather more spin-$\frac{1}{2}$ PGFS are known and the calculations have been pursued to $r+t \leq 10$.

Many applications of the present work will need the detailed information contained in the $g_{r,t}$ and we had hoped to represent these polynomials here. It seems clear, however, that they would take far too much space. We have therefore chosen to compress our data and—in the fashion of the publications of Sykes and collaborators referred to above—quote the high field polynomials L_n†. In the present case these are given by

$$L_n = \sum_{r=0}^{n} g_{r,n-r} \tag{5.1}$$

These are relevant to the special model of Schofield and Bowers (1980, 1981) in which $m_A/S_1 = m_B/S_2$. In this case, in a uniform field $H_A = H_B$, from (1.2), $\mu = \nu$ and (3.5) can be rewritten as

$$\ln \Lambda(\mu, u) = \sum L_n(u)\mu^n. \tag{5.2}$$

The high field polynomials L_n are given in an appendix (in which L_n is written in place of $L_n(u)$).

As mentioned previously, a study, based on the present work, of the shape of the critical isotherm in the mixed spin ferromagnet has already been published (Yousif and Bowers 1983b). This uses the L_n. Other possible areas of application—of the L_n or $g_{r,t}$—include low temperature ferromagnetic critical behaviour and the interrelation between uniform and staggered descriptions of ferrimagnetic critical behaviour (Bowers 1981, Bowers and Schofield 1981, Yousif and Bowers 1983a, Bowers and Yousif 1983). These must clearly form the subject of separate studies.

Appendix. High-field polynomials L_n

Body centred cubic lattice

$$L_1 = \tfrac{1}{2}u^4 + \tfrac{1}{2}u^8$$

$$L_2 = \tfrac{1}{4}u^8 + 4u^{11} - 4u^{12} - \tfrac{1}{4}u^{16}$$

$$L_3 = -\tfrac{1}{3}u^{12} + 18u^{14} - 32u^{15} + 14u^{16} + 14u^{18} - 32u^{19} + 18u^{20} + \tfrac{1}{6}u^{24}$$

$$\begin{aligned} L_4 = {} & \tfrac{1}{8}u^{16} + 56u^{17} - 144u^{18} + 112u^{19} + 2u^{20} + 148u^{21} - 606u^{22} + 660u^{23} - 228u^{24} \\ & + 28u^{25} - 112u^{26} + 144u^{27} - 60u^{28} - \tfrac{1}{8}u^{32} \end{aligned}$$

$$\begin{aligned} L_5 = {} & 133\tfrac{1}{10}u^{20} - 448u^{21} + 534u^{22} + 64u^{23} + 561u^{24} - 5088u^{25} + 8800u^{26} \\ & - 5844u^{27} + 1914u^{28} - 4312u^{29} + 8436u^{30} - 6612u^{31} + 1897u^{32} \\ & - 224u^{33} + 504u^{34} - 480u^{35} + 165u^{36} + \tfrac{1}{10}u^{40} \end{aligned}$$

$$\begin{aligned} L_6 = {} & 252u^{23} - 1034\tfrac{1}{6}u^{24} + 1784u^{25} - 162u^{26} - 476u^{27} - 8098u^{28} + 12\,564u^{29} \\ & + 5326u^{30} - 14\,492u^{31} - 51\,459\tfrac{1}{2}u^{32} + 174\,577\tfrac{7}{20}u^{33} - 211\,110u^{34} \\ & + 118\,680u^{35} - 42\,559\tfrac{7}{20}u^{36} + 53\,976u^{37} - 71\,638u^{38} + 44\,588u^{39} \\ & - 10\,970\tfrac{1}{2}u^{40} + 1008u^{41} - 1680u^{42} + 1320u^{43} - 369u^{44} - \tfrac{1}{12}u^{48} \end{aligned}$$

† Interested readers may obtain the $g_{r,t}$ by writing directly to the authors.

$$L_7 = 410u^{26} - 1908u^{27} + 3796\tfrac{1}{14}u^{28} - 1148u^{29} - 1006u^{30} - 6512u^{31} + 12\,737u^{32} + 20\,108u^{33} - 78\,608u^{34} - 380\,468u^{35} + 1932\,847u^{36} - 3326\,016u^{37} + 2792\,924u^{38} - 1522\,148u^{39} + 1894\,862u^{40} - 3177\,916u^{41} + 2970\,366u^{42} - 1440\,816u^{43} + 477\,378u^{44} - 427\,700u^{45} + 440\,956u^{46} - 231\,092u^{47} + 50\,004u^{48} - 3360u^{49} + 4620^{50} - 3168u^{51} + 858u^{52} + \tfrac{1}{14}u^{56}.$$

Simple cubic lattice

$$L_1 = \tfrac{1}{2}u^3 + \tfrac{1}{2}u^6$$

$$L_2 = \tfrac{1}{4}u^6 + 3u^8 - 3u^9 - \tfrac{1}{4}u^{12}$$

$$L_3 = -\tfrac{1}{6}u^9 + 10\tfrac{1}{2}u^{10} - 18u^{11} + 7\tfrac{1}{2}u^{12} + 7\tfrac{1}{2}u^{13} - 18u^{14} + 10\tfrac{1}{2}u^{15} + \tfrac{1}{6}u^{18}$$

$$L_4 = 25\tfrac{1}{8}u^{12} - 63u^{13} + 55\tfrac{1}{2}u^{14} + 56u^{15} - 256\tfrac{1}{2}u^{16} + 279u^{17} - 86u^{18} - 45u^{19} + 63u^{20} - 28u^{21} - \tfrac{1}{8}u^{24}$$

$$L_5 = 45u^{14} - 143\tfrac{9}{10}u^{15} + 244\tfrac{1}{2}u^{16} + 150u^{17} - 1666\tfrac{1}{2}u^{18} + 2835u^{19} - 1696\tfrac{1}{2}u^{20} - 915u^{21} + 2671\tfrac{1}{2}u^{22} - 2128\tfrac{1}{2}u^{23} + 552u^{24} + 157\tfrac{1}{2}u^{25} - 168u^{26} + 63u^{27} + \tfrac{1}{10}u^{30}$$

$$L_6 = 66u^{16} - 222u^{17} + 721\tfrac{1}{3}u^{18} - 21u^{19} - 6578\tfrac{1}{4}u^{20} + 16\,665u^{21} - 16\,026u^{22} - 7167u^{23} + 41\,175\tfrac{1}{4}u^{24} - 50\,484u^{25} + 24\,195\tfrac{3}{4}u^{26} + 6566u^{27} - 17\,292u^{28} + 11\,133u^{29} - 2564\tfrac{1}{4}u^{30} - 420u^{31} + 378u^{32} - 126u^{33} - \tfrac{1}{12}u^{36}$$

$$L_7 = 94\tfrac{1}{2}u^{18} - 192u^{19} + 1560u^{20} - 1428\tfrac{13}{14}u^{21} - 17\,904u^{22} + 66\,828u^{23} - 98\,327\tfrac{1}{2}u^{24} - 18\,960u^{25} + 345\,273u^{26} - 610\,031\tfrac{1}{2}u^{27} + 441\,945u^{28} + 97\,959u^{29} - 539\,822u^{30} + 533\,739u^{31} - 221\,289u^{32} - 26\,423u^{33} + 82\,480\tfrac{1}{2}u^{34} - 45\,472\tfrac{1}{2}u^{35} + 9551\tfrac{1}{2}u^{36} + 945u^{37} - 756u^{38} + 231u^{39} + \tfrac{1}{14}u^{42}$$

Simple quadratic lattice

$$L_1 = \tfrac{1}{2}u^2 + \tfrac{1}{2}u^4$$

$$L_2 = \tfrac{1}{4}u^4 + 2u^5 - 2u^6 - \tfrac{1}{4}u^8$$

$$L_3 = 4\tfrac{2}{3}u^6 - 8u^7 + 6u^8 - 8u^9 + 5u^{10} + \tfrac{1}{6}u^{12}$$

$$L_4 = 8u^7 - 15\tfrac{7}{8}u^8 + 26u^9 - 69u^{10} + 80u^{11} - 39u^{12} + 20u^{13} - 10u^{14} - \tfrac{1}{8}u^{16}$$

$$L_5 = 11\tfrac{1}{2}u^8 - 14u^9 + 48\tfrac{1}{10}u^{10} - 272u^{11} + 493\tfrac{1}{2}u^{12} - 510u^{13} + 519\tfrac{1}{2}u^{14} - 404u^{15} + 150u^{16} - 40u^{17} + 17\tfrac{1}{2}u^{18} + \tfrac{1}{10}u^{20}$$

$$L_6 = 1u^8 + 16u^9 + 5u^{10} + 32u^{11} - 656\tfrac{1}{6}u^{12} + 1642u^{13} - 2789u^{14} + 4762\tfrac{2}{3}u^{15} - 5738u^{16} + 4300u^{17} - 2620\tfrac{2}{3}u^{18} + 1452u^{19} - 449u^{20} + 70u^{21} - 28u^{22} - \tfrac{1}{12}u^{24}$$

$$L_7=4u^9+25u^{10}+42u^{11}-46u^{12}-1178u^{13}+3326\tfrac{8}{14}u^{14}-8520u^{15}+23\,378\tfrac{1}{2}u^{16}$$
$$-40\,914u^{17}+48\,443u^{18}-48\,686u^{19}+40\,777\tfrac{1}{2}u^{20}-23\,788u^{21}$$
$$+10\,276\tfrac{1}{2}u^{22}-4216u^{23}+1145u^{24}-112u^{25}+42u^{26}+\tfrac{1}{14}u^{28}$$

Honeycomb lattice

$$L_1=\tfrac{1}{2}z^3+\tfrac{1}{2}z^6$$
$$L_2=\tfrac{1}{4}z^6+1\tfrac{1}{2}z^7-1\tfrac{1}{2}z^9-\tfrac{1}{4}z^{12}$$
$$L_3=3z^8-\tfrac{1}{3}z^9-4\tfrac{1}{2}z^{10}+1\tfrac{1}{2}z^{11}+1\tfrac{1}{2}z^{12}-4\tfrac{1}{2}z^{13}+3z^{15}+\tfrac{1}{6}z^{18}$$
$$L_4=3\tfrac{1}{2}z^9+1\tfrac{1}{2}z^{10}-9z^{11}+6\tfrac{1}{8}z^{12}+4\tfrac{1}{2}z^{13}-29\tfrac{1}{4}z^{14}+1\tfrac{1}{2}z^{15}+33z^{16}$$
$$-4\tfrac{1}{2}z^{17}-11\tfrac{1}{4}z^{18}+9z^{19}-5z^{21}-\tfrac{1}{8}z^{24}$$
$$L_5=3z^{10}+6z^{11}-10z^{12}+7\tfrac{1}{2}z^{13}+9z^{14}-93\tfrac{4}{10}z^{15}+7\tfrac{1}{2}z^{16}+160\tfrac{1}{2}z^{17}-63\tfrac{1}{2}z^{18}-97\tfrac{1}{2}z^{19}$$
$$+144z^{20}+15\tfrac{1}{2}z^{21}-130\tfrac{1}{2}z^{22}+9z^{23}+40\tfrac{1}{2}z^{24}-15z^{25}+7\tfrac{1}{2}z^{27}+\tfrac{1}{10}z^{30}$$
$$L_6=1\tfrac{1}{2}z^{11}+15z^{12}-6z^{13}-6z^{14}+19\tfrac{1}{2}z^{15}-189z^{16}-12z^{17}+463\tfrac{1}{3}z^{18}-303z^{19}$$
$$-402\tfrac{3}{4}z^{20}+1029\tfrac{1}{2}z^{21}+56\tfrac{1}{4}z^{22}-1327\tfrac{1}{2}z^{23}+276\tfrac{3}{4}z^{24}+750z^{25}$$
$$-483\tfrac{3}{4}z^{26}-149z^{27}+378z^{28}-15z^{29}-108z^{30}+22\tfrac{1}{2}z^{31}-10\tfrac{1}{2}z^{33}-\tfrac{1}{12}z^{36}$$
$$L_7=\tfrac{1}{2}z^{12}+24z^{13}+13\tfrac{1}{2}z^{14}-46\tfrac{1}{2}z^{15}+43\tfrac{1}{2}z^{16}-250\tfrac{1}{2}z^{17}-241z^{18}+927z^{19}-661\tfrac{1}{2}z^{20}$$
$$-1113\tfrac{6}{14}z^{21}+4183\tfrac{1}{2}z^{22}-37\tfrac{1}{2}z^{23}-7409\tfrac{1}{2}z^{24}+3001\tfrac{1}{2}z^{25}$$
$$+5989\tfrac{1}{2}z^{26}-6602\tfrac{1}{2}z^{27}-1950z^{28}+6984z^{29}-590\tfrac{1}{2}z^{30}-3603z^{31}$$
$$+1282\tfrac{1}{2}z^{32}+712z^{33}-903z^{34}+22\tfrac{1}{2}z^{35}+241\tfrac{1}{2}z^{36}$$
$$-31\tfrac{1}{2}z^{37}+14z^{39}+\tfrac{1}{14}z^{42}$$
$$L_8=1\tfrac{1}{2}z^{13}+28\tfrac{1}{2}z^{14}+55z^{15}-88\tfrac{1}{2}z^{16}+37\tfrac{1}{2}z^{17}-185\tfrac{1}{4}z^{18}-993z^{19}+1262\tfrac{1}{4}z^{20}-142\tfrac{1}{2}z^{21}$$
$$-2689\tfrac{1}{2}z^{22}+10\,938z^{23}+541\tfrac{1}{16}z^{24}-26\,905\tfrac{1}{2}z^{25}+16\,034\tfrac{1}{4}z^{26}$$
$$+28\,460\tfrac{1}{2}z^{27}-48\,530\tfrac{5}{8}z^{28}-10\,206z^{29}+67\,499\tfrac{3}{4}z^{30}-13\,813\tfrac{1}{2}z^{31}$$
$$-49\,437z^{32}+29\,410z^{33}+17\,490\tfrac{3}{4}z^{34}-27\,405z^{35}$$
$$-438\tfrac{1}{8}z^{36}+13\,056z^{37}-2898z^{38}-2485\tfrac{1}{2}z^{39}+1890z^{40}$$
$$-31\tfrac{1}{2}z^{41}-479\tfrac{1}{2}z^{42}+42z^{43}-18z^{45}-\tfrac{1}{16}z^{48}$$
$$L_9=\tfrac{1}{2}z^{12}+4\tfrac{1}{2}z^{14}+27z^{15}+121\tfrac{1}{2}z^{16}-82\tfrac{1}{2}z^{17}-89z^{18}+60z^{19}-2431\tfrac{1}{2}z^{20}+601\tfrac{1}{2}z^{21}$$
$$+3445\tfrac{1}{2}z^{22}-6085\tfrac{1}{2}z^{23}+18\,896\tfrac{1}{2}z^{24}+10\,615\tfrac{1}{2}z^{25}-69\,918z^{26}$$
$$+45\,785\tfrac{8}{9}z^{27}+96\,453z^{28}-222\,901\tfrac{1}{2}z^{29}-30\,192\tfrac{1}{2}z^{30}+405\,328\tfrac{1}{2}z^{31}$$
$$-147\,952\tfrac{1}{2}z^{32}-385\,619\tfrac{1}{2}z^{33}+357\,246z^{34}+179\,641\tfrac{1}{2}z^{35}$$
$$-422\,325\tfrac{1}{2}z^{36}+15\,946\tfrac{1}{2}z^{37}+282\,162z^{38}$$
$$-99\,037\tfrac{1}{2}z^{39}-98\,323\tfrac{1}{2}z^{40}+87\,645z^{41}+9783z^{42}-39\,078z^{43}+5838z^{44}$$
$$+7142\tfrac{1}{2}z^{45}-3591z^{46}+42z^{47}+873z^{48}-54z^{49}+22\tfrac{1}{2}z^{51}+\tfrac{1}{18}z^{54}$$

$$
\begin{aligned}
L_{10} = {} & 1\tfrac{1}{2}z^{13}+1\tfrac{1}{2}z^{14}+6z^{15}+30z^{16}+193\tfrac{1}{2}z^{17}+59\tfrac{1}{2}z^{18}-433\tfrac{1}{2}z^{19}+384z^{20}-4034z^{21} \\
& -3319\tfrac{1}{2}z^{22}+12\,019\tfrac{1}{2}z^{23}-10\,155\tfrac{1}{2}z^{24}+17\,625z^{25}+59\,574\tfrac{3}{4}z^{26} \\
& -134\,618\tfrac{1}{2}z^{27}+45\,783z^{28}+270\,449\tfrac{19}{20}z^{29}-689\,570\tfrac{2}{10}z^{30} \\
& -109\,838\tfrac{19}{20}z^{31}+1681\,737\tfrac{3}{4}z^{32}-868\,645z^{33}-2000\,297\tfrac{1}{4}z^{34} \\
& +2655\,106\tfrac{8}{10}z^{35}+1044\,865\tfrac{3}{4}z^{36}-3931\,809z^{37}+591\,453\tfrac{3}{4}z^{38} \\
& +3382\,189z^{39}-1848\,420z^{40}-1626\,027z^{41}+2013\,867\tfrac{7}{10}z^{42} \\
& +237\,351\tfrac{1}{2}z^{43}-1248\,462\tfrac{13}{20}z^{44}+257\,770\tfrac{2}{10}z^{45}+419\,365\tfrac{9}{20}z^{46} \\
& -241\,083z^{47}-50\,960\tfrac{1}{4}z^{48}+101\,745z^{49}-10\,786\tfrac{1}{2}z^{50}-17\,905z^{51} \\
& +6336z^{52}-54z^{53}-1487\tfrac{1}{4}z^{54}+67\tfrac{1}{2}z^{55}-27\tfrac{1}{2}z^{57}-\tfrac{1}{20}z^{60}.
\end{aligned}
$$

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